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A strong log-concavity property for measures on Boolean algebras ☆

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ABSTRACT

We introduce the antipodal pairs property for probability measures on finite Boolean algebras and prove that conditional versions imply strong forms of log-concavity. We give several applications of this fact, including improvements of some results of Wagner, a new proof of a theorem of Liggett stating that ultra-log-concavity of sequences is preserved by convolutions, and some progress on a well-known log-concavity conjecture of J. Mason.

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1. Introduction

1.1. Log-concavity and the antipodal pairs property

Before stating our main result, we fix some notation and recall some definitions. Given a finite set S, denote by $\mathcal{M} = \mathcal{M}_S$ the set of probability measures on $\Omega = \Omega_S = \{0,1\}^S$. (In this setting, a probability measure on Ω is a nonnegative function μ on Ω with $\sum_{\eta \in \Omega} \mu(\eta) = 1$; we often just write "measure" in place of "probability measure".) As a default we take $S = [n] = \{1, \dots, n\}$ (which for us is simply a generic n-set), using Ω for $\Omega_{[n]}$ and $\mathcal{M}$ or $\mathcal{M}_n$ for $\mathcal{M}_{[n]}$. We will occasionally identify Ω with the Boolean algebra $2^{[n]}$ (the collection of subsets of $[n]$ ordered by inclusion) in the natural way (namely, identifying a set with its indicator).

We will be interested in several properties of measures that are preserved by the operation of *conditioning*, which for us always means fixing the values of some variables (this specification is always assumed to have positive probability); thus a measure obtained from $\mu \in \mathcal{M}$ by conditioning is one of the form $\mu(\cdot|\eta_i = \xi_i \ \forall i \in I)$ (which we regard as a measure on $\Omega_{[n]\setminus I}$) for some $I \subseteq [n]$ and

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$\xi \in \{0,1\}^I$. (If we think of Ω as $2^{[n]}$, then conditioning amounts to restricting our measure to some interval $[J,K]$ of $2^{[n]}$ (and normalizing).)

Recall that a sequence $a=(a_0,\dots,a_n)$ of real numbers (here always nonnegative) is *unimodal* if there is some $k\in\{0,\dots,n\}$ for which $a_0\leqslant a_1\leqslant\cdots\leqslant a_k\geqslant\cdots\geqslant a_n$, and *log-concave* (LC) if $a_i^2\geqslant a_{i-1}a_{i+1}$ for $1\leqslant i\leqslant n-1$. Of course a nonnegative LC sequence with no internal zeros is unimodal (where "no internal zeros" means $\{i\colon a_i\neq 0\}$ is an interval). Following [16] we say a (as above) is *ultra-log-concave* (ULC) if the sequence $(a_i/\binom{n}{i})_{i=0}^n$ is log-concave and has no internal zeros. We also say $\mu\in\mathcal{M}$ is ULC if its *rank sequence*, $(\mu(|\eta|=i))_{i=0}^n$, is ULC (where $|\eta|=\sum\eta_i$). We define "μ is LC" and "μ is unimodal" similarly, except that for the former we add the requirement that the rank sequence have no internal zeros.

For $\mu\in\mathcal{M}$ set

$$\alpha_i(\mu)=\binom{n}{i}^{-1}\sum\bigl\{\mu(\eta)\mu(\underline{1}-\eta)\colon\ \eta\in\Omega,\ |\eta|=i\bigr\} \tag{1}$$

(where $\underline{1}=(1,\dots,1)$). Say that $\mu\in\mathcal{M}_{2k}$ has the *antipodal pairs property* (APP) if $\alpha_k(\mu)\geqslant\alpha_{k-1}(\mu)$, and that $\mu\in\mathcal{M}$ has the *conditional antipodal pairs property* (CAPP) if every measure obtained from μ by a conditioning that fixes the values of some $n-2k$ variables (for some k) has the APP.

Example 1. Consider the measure μ on $\{0,1\}^4$ with

$$\mu(\eta)\propto\begin{cases}1 & \text{if } |\eta|=0,\\ 2 & \text{if } |\eta|=1 \text{ and } \eta\neq(1,0,0,0),\\ \varepsilon & \text{if } \eta=(1,0,0,0),\\ 4 & \text{if } |\eta|=2,\\ 3 & \text{if } |\eta|=3,\\ 2 & \text{if } |\eta|=4\end{cases}$$

(meaning, as usual, that the left side is the right side multiplied by the appropriate normalizing constant). It's straightforward to verify that μ has the CAPP when $2\leqslant\varepsilon\leqslant 16/3$, has the APP but not the CAPP when $0\leqslant\varepsilon<2$ or $16/3<\varepsilon\leqslant 46/3$, and otherwise has neither property.

See also Theorem 8 below for some additional examples of measures with the CAPP.

Our main result is

Theorem 2. *For measures without internal zeros in their rank sequence, the CAPP implies ULC.*

A somewhat more general version of Theorem 2 is stated and proved in Section 2. In the rest of this introduction we provide a little context and sketch some consequences to be established in later sections.

1.2. Negative dependence properties

We need to briefly review a few negative dependence notions; for much more on this see e.g. [1,2,10,16,18]. Recall that events $\mathcal{A}$, $\mathcal{B}$ in a probability space are *negatively correlated*—we write $\mathcal{A}\downarrow\mathcal{B}$—if $\Pr(\mathcal{A}\mathcal{B})\leqslant\Pr(\mathcal{A})\Pr(\mathcal{B})$. We say μ has *negative correlations* (or *is NC*) if $\eta_i\downarrow\eta_j$ (that is, $\{\eta_i=1\}\downarrow\{\eta_j=1\}$) whenever $i\neq j$. A stronger property is obtained by requiring NC for every measure $W\circ\mu$ of the form

$$W\circ\mu(\eta)\propto\mu(\eta)\prod W_i^{\eta_i}$$

with $W=(W_1,\dots,W_n)\in\mathbf{R}_+^n$; we say $W\circ\mu$ is obtained from μ by *imposing an external field* and (following [2,18]) say μ has the *Rayleigh property* if every measure gotten from μ by imposing an external field is NC.

Theorem 2 was discovered during attempts to prove a sequence of conjectures of Pemantle stating that various negative dependence properties, including the Rayleigh property, imply ULC; see

[16, Conjecture 4]. That Rayleigh implies ULC was also conjectured by Wagner [18]. As it happens, even the weakest version of Pemantle's conjecture is false; see [1,10] for counterexamples and further discussion. Still, Theorem 2 does turn out to be helpful in establishing ULC in some other settings, which we now indicate.

A first, easy consequence is improvement of some of the results of [17], for which we need to recall some terminology from that paper. Given a positive integer k and positive real number λ, say that μ satisfies λ-Ray[k] if every measure ν gotten from μ by imposing an external field and then projecting onto a set S of $2k$ variables satisfies

$$\sum\bigl\{\nu(\eta)\nu(\underline{1}-\eta)\colon\ \eta\in\Omega,\ |\eta|=k\bigr\}\geqslant\lambda\sum\bigl\{\nu(\eta)\nu(\underline{1}-\eta)\colon\ \eta\in\Omega,\ |\eta|=k-1\bigr\}. \tag{2}$$

(For the (standard) definition of projection, see Section 3.) With the notation of (1), the above condition is

$$\alpha_k(\nu)\geqslant\frac{\lambda k}{k+1}\alpha_{k-1}(\nu);$$

thus $(1+1/k)$-Ray[k] says that each ν as above has the APP. (As observed by Wagner [17]—see his Proposition 4.6—$\lambda=(1+1/k)$ is an "especially natural strength for these conditions"; there as here, this is essentially because $1+1/k$ is the ratio of the numbers of summands on the two sides of (2).) Note also that 2-Ray[1] is precisely the Rayleigh property.

Say that $\mu\in\mathcal{M}$ is BLC[m] if every measure gotten from μ by imposing an external field and then projecting onto a set of size at most m is ULC (the acronym is for "binomial log-concavity"), and BLC if it is BLC[m] for all m. In [2] and [10], BLC[m] is called LC[m]. Wagner proved

Theorem 3. *(See [17, Theorem 4.3].) If a measure satisfies* 2-Ray[1] *and* $(1+1/k)^2$-Ray[k] *for all* $2\leqslant k\leqslant m$*, then it is* BLC[$2m+1$].

(In [17] this is stated only for uniform measure on the bases of a matroid, but the proof is valid for general measures in $\mathcal{M}$.) Theorem 2 implies the following strengthening (see Section 3).

Corollary 4. *If a measure satisfies* $(1+1/k)$-Ray[k] *for all* $1\leqslant k\leqslant m$*, then it is* BLC[$2m+1$].

Using Corollary 4 in place of Theorem 3 improves Corollary 4.5(b) and Theorem 5.2 of [17] by substituting BLC for the weaker property $\sqrt{\text{BLC}}$; see [17] for definitions and statements.

1.3. Ultra-log-concave sequences

It is easy to see that if $\mu\in\mathcal{M}_S$ and $\nu\in\mathcal{M}_T$ are Rayleigh then the product measure $\mu\times\nu$ (given by $\mu\times\nu(\xi,\eta)=\mu(\xi)\nu(\eta)$ for $(\xi,\eta)\in\{0,1\}^S\times\{0,1\}^T$) is also Rayleigh. Note that the rank sequence of $\mu\times\nu$ is the convolution of the rank sequences for μ and ν. One consequence of the (false) conjecture mentioned earlier, that Rayleigh measures are ULC, would have been that the convolution of two ULC sequences is ULC or, equivalently, that the product of two ULC measures is ULC. (The implication follows from a result of Pemantle [16, Theorem 2.7] stating that for *exchangeable* measures (those for which $\mu(\eta)$ depends only on $|\eta|$) the properties Rayleigh and ULC coincide.) Surprisingly—given that the analogous statement for ordinary log-concavity is fairly trivial—preservation of ULC under convolution turns out not to be so obvious; it was conjectured by Pemantle [16] (motivated by the preceding considerations) and proved by Liggett:

Theorem 5. *(See [11, Theorem 2].) The convolution of two ULC sequences is ULC.*

In Section 4 we derive this from Theorem 2 and also discuss a potentially interesting strengthening of ULC for measures that is again preserved by products. In contrast Liggett's original proof is wholly elementary (ours is not, since we will use Bose–Mesner algebras) but ingenious and (to us) less intuitive, while the elegant recent proof of Gurvits [7] is shorter but uses much heavier machinery (mixed volumes and the Alexandrov–Fenchel inequalities).

1.4. Mason's conjecture

For the purposes of this introduction we regard a matroid as a collection $\mathcal{I}$ of independent subsets of some ground set E. We will not go into matroid definitions; see e.g. [20] or [15]. Prototypes are the collection of (edge sets of) forests of a graph (with edge set E)—this is a *graphic* matroid—and (as it turns out, more generally) the collection of linearly independent subsets of some finite subset E of some (not necessarily finite) vector space; for present purposes not too much is lost by thinking only of graphic matroids.

We are interested in the *independence numbers* of $\mathcal{I}$, that is, the numbers

$$a_k = a_k(\mathcal{I}) = \left|\left\{I \in \mathcal{I}\colon\ |I| = k\right\}\right|, \quad k = 0, \ldots, n,$$

for which a celebrated conjecture of Mason [14] says

Conjecture 6. *For any matroid $\mathcal{I}$ on a ground set of size n, the sequence $a = a(\mathcal{I}) = (a_0, \ldots, a_n)$ is ULC.*

(Note that a will typically end with some 0's, and also that in the graphic case n counts edges, not vertices.) Of course one can weaken Conjecture 6 by asking for LC or unimodality in place of ULC. In fact Mason also stated the LC version, and unimodality, first suggested by Welsh [19], was the original conjecture in this direction (and even this, even for graphic matroids, remains open).

From the present viewpoint, Mason's conjecture says that uniform measure on $\mathcal{I}$ (regarded in the usual way as a subset of $\{0, 1\}^E$) is ULC. (When $\mathcal{I}$ is graphic such a measure is a *uniform spanning forest* (USF) measure ("spanning" because we think of a member of $\mathcal{I}$ as a subgraph that includes all vertices).) In particular, according to Theorem 2, Mason's conjecture would follow from

Conjecture 7. *Uniform measure on the independent sets of a matroid has the CAPP.*

See also the remark following Corollary 13 for a possible strengthening.

Of course here it's enough to show APP, since each conditional measure is just uniform measure on the independent sets of some minor. Also, note that Mason's conjecture for graphic matroids would have followed from the (false) conjecture of Pemantle [16] and Wagner [18] that Rayleigh measures are ULC, if it could be shown that, as conjectured in [9] (see [6,10,18] for more on this), USF measures are Rayleigh.

Though probably not for lack of effort, progress on Mason's conjecture has been fairly modest. Dowling [4] proved that for each $\mathcal{I}$ the sequence $(a_0, \ldots, a_8)$ is LC; Mahoney [13] proved that for graphic matroids corresponding to *outerplanar* graphs, the full sequence of independence numbers is LC; and Hamidoune and Salaün [8] proved that for any matroid on a ground set of size n the sequence $(a_i/\binom{n}{i})_{i=0}^4$ is LC, i.e. the sequence (a_i) is "ULC up to 4".

Here we adapt one of Dowling's arguments to prove Conjecture 7 for small matroids:

Theorem 8. *For every matroid on a ground set of size at most* 11, *uniform measure on independent sets has the CAPP.*

This is proved in Section 5. Combined with Theorem 2 (for (a)) or the more general Theorem 10 below (for (b)) it gives

Theorem 9. (a) *Every matroid on a ground set of size at most* 11 *satisfies Conjecture 6.* (b) *For any matroid on a ground set of size n with independence numbers a_i, the sequence $\left(a_i/\binom{n}{i}\right)_{i=0}^{6}$ is LC (a.k.a. the sequence (a_i) is "ULC up to* 6*").*

2. Proof of Theorem 2

We actually prove a more general result that will be needed in Section 5.

Theorem 10. *Suppose $\mu \in \mathcal{M}$ has the property that, for every $k \in [t]$, every measure gotten from μ by conditioning on the values of $n-2k$ coordinates has the APP. Then the sequences $(\mu(|\eta|=i)/\binom{n}{i})_{i=0}^{t+1}$ and $(\mu(|\eta|=i)/\binom{n}{i})_{i=n-t-1}^{n}$ are LC.*

For the rest of this section it will be convenient to treat Ω as $2^{[n]}$, so that (1) becomes

$$\alpha_i(\mu) = \binom{n}{i}^{-1} \sum \left\{ \mu(X)\mu\big([n]\setminus X\big)\colon\ X \in \binom{[n]}{i} \right\}$$

(where $\binom{[n]}{i} = \{X \subseteq [n]\colon\ |X| = i\}$).

In outline the proof goes as follows. We rewrite the desired inequalities (5) in the form

$$l(n-l) \sum_{X\in\binom{[n]}{l}} \sum_{Y\in\binom{[n]}{l}} \mu(X)\mu(Y) \geqslant (l+1)(n-l+1) \sum_{X\in\binom{[n]}{l-1}} \sum_{Y\in\binom{[n]}{l+1}} \mu(X)\mu(Y),$$

group terms on each side according to the value of $(X\cap Y, X\cup Y)$, and apply our CAPP assumptions to produce the "local" inequalities (9). These imply sufficiency (for our purposes) of the inequality (10), and the rest of the proof involves showing that (10) can be gotten from Lemma 11, a (probably not new) assertion on positive-semidefiniteness in the Bose–Mesner algebra of the Johnson scheme in which we are working.

Before proceeding we need to recall some properties of the Johnson scheme; this material (up to (4)) is taken from chapter 30 of [12]. Fix positive integers n and l with $l \leqslant n/2$, let $\mathfrak{X} = \binom{[n]}{l}$, and, for $i = 0, 1, \dots, l$, let A_i be the $\mathfrak{X}\times\mathfrak{X}$ adjacency matrix of ith associates, *viz.*

$$A_i(X,Y) = \begin{cases} 1 & \text{if } |X\cap Y| = l-i, \\ 0 & \text{otherwise.} \end{cases}$$

We write elements of $\mathbf{R}^{\mathfrak{X}}$ as row vectors. For $T \subseteq [n]$ with $|T| \leqslant l$, let e_T be the vector in $\mathbf{R}^{\mathfrak{X}}$ with

$$e_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise,} \end{cases}$$

and let U_i be the span of $\{e_T\colon\ T \in \binom{[n]}{i}\}$. Then $\dim U_i = \binom{n}{i}$ and $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l = \mathbf{R}^{\mathfrak{X}}$. Set $V_0 = U_0$ and $V_i = U_i \cap U_{i-1}^{\perp}$ for $i = 1, 2, \dots, l$, and let E_i be the projection of $\mathbf{R}^{\mathfrak{X}}$ onto V_i. Then

$$\mathbf{R}^{\mathfrak{X}} = V_0 \oplus V_1 \oplus \cdots \oplus V_l$$

is an orthogonal decomposition,

$$E_iE_j = \begin{cases} E_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and

$$E_0 + E_1 + \cdots + E_l = I.$$

Note that V_0 consists of the constant vectors. The span, $\mathfrak{A}$, of $A_0, \dots, A_l$ is an algebra under matrix multiplication (the *Bose–Mesner algebra*). The set of matrices $\{E_0, E_1, \dots, E_l\}$ is also a basis for $\mathfrak{A}$, with

$$A_i = \sum_{j=0}^{l} P_i(j) E_j \quad (i = 0, 1, \dots, l), \tag{3}$$

where

$$P_i(j) = \sum_{k=0}^{i} (-1)^{i-k} \binom{l-k}{i-k} \binom{l-j}{k} \binom{n-l+k-j}{k}. \tag{4}$$

The next lemma is presumably well known.

Lemma 11. *For any $\gamma_0, \gamma_1, \ldots, \gamma_l \in \mathbf{R}$, the $\mathfrak{X} \times \mathfrak{X}$ real symmetric matrix $M = \sum_{i=0}^{l} \gamma_i A_i$ is positive semidefinite if and only if $\sum_{i=0}^{l} \gamma_i P_i(j) \geqslant 0$ for $j = 0, 1, \ldots, l$.*

Proof. Since $M = \sum_{j=0}^{l} \sum_{i=0}^{l} \gamma_i P_i(j) E_j$ and E_j is the orthogonal projection of $\mathbf{R}^{\mathfrak{X}}$ onto V_j, the eigenvalues of M are $\{\sum_{i=0}^{l} \gamma_i P_i(j)\colon\ j = 0, 1, \ldots, l\}$. $\square$

Remark. It is not hard to show, using some additional properties of the Johnson scheme, that the condition appearing in Lemma 11 is equivalent to the statement that the vector $(\binom{l}{i}\binom{n-l}{l-i}\gamma_i)_{i=0}^{l}$ satisfies *Delsarte's inequalities* ([3] or [12, p. 416]).

We also need the following technical result, an easy consequence of e.g. equation (5.41) in [5]:

Proposition 12. *For all positive integers M, N and real numbers a, b,*

$$\sum_{t=0}^{N} (-1)^t \frac{at+b}{t+M} \binom{N}{t} = \left(\frac{b}{M} - a\right) \binom{M+N}{M}^{-1}.$$

Proof of Theorem 10. Let μ be a measure on $2^{[n]}$ satisfying the hypotheses of the theorem, with rank sequence $(a_i)_{i=0}^{n}$. Our goal is to show

$$l(n-l)a_l^2 \geqslant (l+1)(n-l+1)a_{l-1}a_{l+1} \tag{5}$$

for $l \in \{1, \ldots, t\} \cup \{n-t, \ldots, n-1\}$; but, since $\mu' \in \mathcal{M}$ given by $\mu'(X) = \mu([n] \setminus X)$ again satisfies the hypotheses of Theorem 10 and has rank sequence $(a_{n-i})_{i=0}^{n}$, it suffices to prove (5) when $l \leqslant \min\{t, n/2\}$. To this end, fix such an l and set

$$Z_{j,k}^{i} = \sum \left\{\mu(X)\mu(Y)\colon\ (X, Y) \in \binom{[n]}{j} \times \binom{[n]}{k} \text{ and } |X \cap Y| = i\right\};$$

with this notation, (5) is

$$l(n-l) \sum_{i=0}^{l} Z_{l,l}^{i} \geqslant (l+1)(n-l+1) \sum_{i=0}^{l-1} Z_{l-1,l+1}^{i}. \tag{6}$$

For each $i \in \{0, 1, \ldots, l-1\}$ and $I \subseteq J \subseteq [n]$ with $|I| = i$ and $|J| = 2l - i$, let $\mu_{I,J} \in \mathcal{M}_{J \setminus I}$ be the conditional measure with

$$\mu_{I,J}(X) \propto \mu(X \cup I) \quad (X \subseteq J \setminus I)$$

(or $\mu_{I,J} \equiv 0$ if $\mu([I, J]) = 0$). By hypothesis (or trivially if $\mu_{I,J} \equiv 0$) $\mu_{I,J}$ has the APP, i.e.

$$\alpha_{l-i}(\mu_{I,J}) \geqslant \alpha_{l-i-1}(\mu_{I,J}). \tag{7}$$

With

$$Z_{j,k}(I, J) = \sum \left\{\mu(X)\mu(Y)\colon\ (X, Y) \in \binom{[n]}{j} \times \binom{[n]}{k},\ X \cup Y = J, \text{ and } X \cap Y = I\right\},$$

we have

$$\alpha_j(\mu_{I,J}) = \left(\sum_{I \subseteq X \subseteq J} \mu(X)\right)^{-1} \binom{2l-2i}{j}^{-1} Z_{i+j,2l-i-j}(I, J),$$

and (7) becomes

$$\frac{l-i}{l-i+1} Z_{l,l}(I, J) \geqslant Z_{l-1,l+1}(I, J). \tag{8}$$

Summing (8) over I, J with $|I| = i$ and $|J| = 2l - i$ gives

$$\frac{l-i}{l-i+1} Z^i_{l,l} \geqslant Z^i_{l-1,l+1} \tag{9}$$

(since each pair (X, Y) contributing to $Z^i_{l,l}$ contributes to the left side of (8) for exactly one choice of (I, J), and similarly for pairs contributing to $Z^i_{l-1,l+1}$). Replacing each $Z^i_{l-1,l+1}$ in (6) by the (corresponding) left side of (9), we find that it is enough to show

$$\sum_{i=0}^{l} \beta_i Z^i_{l,l} \geqslant 0, \tag{10}$$

where

$$\beta_i = \frac{i(n+1) - l(l+1)}{l-i+1}.$$

In fact, we will show that (10) holds for *every* $\mu \in \mathcal{M}$. Let $\psi = \psi_\mu$ be the vector in $\mathbf{R}^{\mathfrak{X}}$ with $\psi(X) = \mu(X)$ for $X \in \binom{[n]}{l}$, and recall the matrices A_i defined before Lemma 11. Since

$$\psi A_i \psi^T = Z^{l-i}_{l,l},$$

the left side of (10) is

$$\psi \left(\sum_{i=0}^{l} \beta_{l-i} A_i \right) \psi^T \tag{11}$$

and (10) will follow from Lemma 11 once we show

$$\sum_{i=0}^{l} \beta_{l-i} P_i(j) \geqslant 0 \tag{12}$$

for $j = 0, 1, \ldots, l$.

Fix $j \in [l]$. (We deal with the case $j = 0$ separately below.) The left side of (12) is

$$\sum_{i=0}^{l} \frac{(l-i)(n+1) - l(l+1)}{i+1} \sum_{k=0}^{i} (-1)^{i-k} \binom{l-k}{i-k} \binom{l-j}{k} \binom{n-l+k-j}{k}, \tag{13}$$

which we want to show is nonnegative. Interchanging the order of summation and making the substitution $t = i - k$, we may rewrite (13) as

$$\sum_{k=0}^{l-j} \binom{l-j}{k} \binom{n-l+k-j}{k} \sum_{t=0}^{l-k} (-1)^t \frac{(l-t-k)(n+1) - l(l+1)}{t+k+1} \binom{l-k}{t}. \tag{14}$$

It is thus enough to show

$$\sum_{t=0}^{l-k} (-1)^t \frac{(l-t-k)(n+1) - l(l+1)}{t+k+1} \binom{l-k}{t} \geqslant 0 \tag{15}$$

whenever $k \leqslant l - 1$. But Proposition 12, with $N = l - k$, $M = k + 1$, $a = -(n + 1)$, and $b = N(n + 1) - l(l + 1)$, says that the left side of (15) is

$$\frac{(n-l+1)(l+1)}{k+1} \binom{l+1}{k+1}^{-1},$$

which is positive since $l \leqslant n/2$.

Finally we show that when $j = 0$, (12) holds with equality. To see this, notice that when μ is uniform measure on $2^{[n]}$, we have equality in (6) and (9), and consequently (10), from which it follows (see (11)) that

$$\psi\left(\sum_{i=0}^{l}\beta_{l-i}A_i\right)\psi^T = 0. \tag{16}$$

But, since $\psi E_j\psi^T$ is $2^{-2n}\binom{n}{l}$ if $j = 0$ and zero otherwise, the left side of (16) is (by (3))

$$2^{-2n}\binom{n}{l}\sum_{i=0}^{l}\beta_{l-i}P_i(0),$$

which gives the promised equality in (12). □

3. Proof of Corollary 4

In this short section we use Theorem 2 to prove Corollary 4. Recall that the *projection* of $\mu \in \mathcal{M}_n$ on $J \subseteq [n]$ is the measure μ' on $\{0,1\}^J$ obtained by integrating out the variables of $[n] \setminus J$; that is,

$$\mu'(\xi) = \sum\left\{\mu(\eta)\colon\ \eta \in \Omega, \eta_i = \xi_i\ \forall i \in J\right\} \quad \left(\xi \in \{0,1\}^J\right).$$

Proof of Corollary 4. The statement is: if $\mu \in \mathcal{M}$ satisfies $(1 + 1/k)$-Ray$[k]$ for all $k \in [m]$, $T \subseteq [n]$, $|T| \leqslant 2m + 1$, and $\nu \in \mathcal{M}_T$ is obtained from μ by imposing an external field and projecting on T, then ν is ULC. By Theorem 2, it suffices to show ν has the CAPP. (We should also show that ν's rank sequence has no internal zeros, but this follows immediately from [18, Proposition 4.7(a)].) Any measure gotten from ν by conditioning on the values of the variables in some set $T \setminus S$ is the limit of a sequence of measures, each gotten from μ by imposing an external field and projecting on S; the CAPP for ν thus follows from our assumption on μ. □

4. Convolution of ULC sequences

In this section we define a property of measures which is stronger than ULC, prove it is preserved by products, and show that this implies Theorem 5.

We begin with some definitions. With $\alpha_i(\mu)$ as in (1), say $\mu \in \mathcal{M}$ is *antipodal pairs unimodal* (APU) if the sequence $(\alpha_i(\mu))_{i=0}^n$ is unimodal (since $\alpha_i(\mu) = \alpha_{n-i}(\mu)$, this means $\alpha_0(\mu) \leqslant \cdots \leqslant \alpha_{\lfloor n/2\rfloor}(\mu) = \alpha_{\lceil n/2\rceil}(\mu) \geqslant \cdots \geqslant \alpha_n(\mu)$), and say μ is *conditionally antipodal pairs unimodal* (CAPU) if every measure obtained from μ by conditioning is APU. Since CAPU trivially implies the CAPP, Theorem 2 gives

Corollary 13. *Every CAPU measure is ULC.*

(As far as we know, Conjecture 7 can be strengthened by replacing "CAPP" with "CAPU".) We will show

Theorem 14. (a) *The product of two APU measures is APU.* (b) *The product of two CAPU measures is CAPU.*

Before giving the proof of Theorem 14, we show that it implies Theorem 5. Recall that $\mu \in \mathcal{M}$ is *exchangeable* if $\mu(\eta)$ depends only on $|\eta| = \sum\eta_i$.

Lemma 15. *For exchangeable measures, the properties ULC and CAPU are equivalent.*

Remark. Pemantle [16, Theorem 2.7] shows that for exchangeable measures, ULC, Rayleigh, and several other negative dependence properties coincide. Lemma 15 adds CAPU to this list.

Proof of Lemma 15. By Corollary 13, we need only show that every exchangeable ULC measure is CAPU; in fact, since conditioning preserves both exchangeability and ULC, it suffices to prove that an exchangeable ULC measure is APU. But if $\mu \in \mathcal{M}$ is exchangeable with rank sequence $(a_0, \dots, a_n)$, then

$$\alpha_i(\mu) = a_i a_{n-i} \binom{n}{i}^{-1} \binom{n}{n-i}^{-1}$$

so that log-concavity of (and absence of internal zeros in) $(a_i/\binom{n}{i})_{i=0}^n$ implies unimodality of $(\alpha_i(\mu))_{i=0}^n$. $\square$

Proof of Theorem 5.. Given ULC sequences $a = (a_0, \dots, a_n)$ and $b = (b_0, \dots, b_m)$, let $\mu \in \mathcal{M}_{[n]}$ and $\nu \in \mathcal{M}_{\{n+1,\dots,n+m\}}$ be the corresponding exchangeable measures; that is,

$$\mu(\eta) = \frac{a_{|\eta|}}{\binom{n}{|\eta|}} \quad \text{and} \quad \nu(\eta) = \frac{b_{|\eta|}}{\binom{m}{|\eta|}}.$$

By Lemma 15, μ and ν are CAPU, so that Theorem 14(b) and Corollary 13 give ULC for $\mu \times \nu \in \mathcal{M}_{[n+m]}$, completing the proof (since the rank sequence of $\mu \times \nu$ is the convolution of a and b). $\square$

Remark. Following [11], say an infinite nonnegative sequence $(a_0, a_1, \dots)$ is ULC[∞] if there are no internal zeros and $a_i^2 \geqslant \frac{i+1}{i} a_{i-1} a_{i+1}$ for $i \geqslant 1$. The proof of Theorem 5 given in [11] allows one or both sequences to be ULC[∞], but an easy limiting argument suffices to get this more general statement from the finite version proved here.

Proof of Theorem 14. Notice that (a) implies (b), since any measure gotten from $\mu \times \nu$ by conditioning is the product of measures obtained from μ and ν by conditioning.

Call a nonnegative sequence $(p_0, \dots, p_s)$ *symmetric* if $p_i = p_{s-i}$ for $i = 0, \dots, s$ and *ultra-unimodal* if $(p_i/\binom{s}{i})_{i=0}^s$ is unimodal. Let $\mu \in \mathcal{M}_{[n]}$ and $\nu \in \mathcal{M}_{\{n+1,\dots,n+m\}}$ be APU. Then $(\binom{n}{i}\alpha_i(\mu))_{i=0}^n$ and $(\binom{m}{i}\alpha_i(\nu))_{i=0}^m$ are symmetric and ultra-unimodal, and we want to say that their convolution, $(\binom{n+m}{k}\alpha_k(\mu \times \nu))_{k=0}^{n+m}$ is ultra-unimodal. So we will be done if we show

Lemma 16. *The convolution of two symmetric ultra-unimodal sequences is ultra-unimodal*

(and symmetric). It's easy to see that Lemma 16 is not true without the symmetry assumption.

Proof. Since every symmetric ultra-unimodal sequence $(p_0, \dots, p_s)$ is a positive linear combination of sequences of the form $(\binom{s}{i}\mathbf{1}_{\{k \leqslant i \leqslant s-k\}})_{i=0}^s$ (and since convolution is bilinear), it suffices to prove that the convolution of $(\binom{s}{i}\mathbf{1}_{\{k \leqslant i \leqslant s-k\}})_{i=0}^s$ and $(\binom{t}{i}\mathbf{1}_{\{l \leqslant i \leqslant t-l\}})_{i=0}^t$ is ultra-unimodal for all k, l, s, t with $k \leqslant s/2$ and $l \leqslant t/2$. (Of course this is also implied by Theorem 5.)

To see this set (for k, l, s, t as above)

$$f_j = \binom{s+t}{j}^{-1} \sum_i \binom{s}{i} \mathbf{1}_{\{k \leqslant i \leqslant s-k\}} \binom{t}{j-i} \mathbf{1}_{\{l \leqslant j-i \leqslant t-l\}};$$

so we should show

$$f_j \leqslant f_{j+1} \quad \text{for all } j < (s+t)/2. \tag{17}$$

It's convenient to work with the natural interpretation of f_j as a probability. Let S and T be disjoint sets with $|S| = s$ and $|T| = t$, and let

$$Q = \{Z \subseteq S \cup T\colon k \leqslant |Z \cap S| \leqslant s-k, l \leqslant |Z \cap T| \leqslant t-l\}.$$

Then $f_j = \Pr(X_j \in Q)$, where X_j is chosen uniformly from $\binom{S\cup T}{j}$. To prove (17), we consider the usual coupling of $X = X_j$ and $Y = X_{j+1}$; namely, choose X uniformly from $\binom{S\cup T}{j}$ and y uniformly from $(S \cup T) \setminus X$, and set $Y = X \cup \{y\}$. We have

$$f_{j+1} - f_j = \Pr(X \notin Q, Y \in Q) - \Pr(X \in Q, Y \notin Q),$$

so should show that the right side is nonnegative.

We may assume $j \geqslant k + l$, since otherwise we cannot have $X \in Q$. Then $\{X \notin Q, Y \in Q\}$ occurs if and only if either (i) $|X \cap S| = k - 1$, $y \in S$, and $j - k + 1 \leqslant t - l$, or (ii) $|X \cap T| = l - 1$, $y \in T$, and $j - l + 1 \leqslant s - k$; thus,

$$\Pr(X \notin Q, Y \in Q) = \binom{s}{k-1}\binom{t}{j-k+1}\frac{s-k+1}{s+t-j}\mathbf{1}_{\{j-k+1\leqslant t-l\}} + \binom{t}{l-1}\binom{s}{j-l+1}\frac{t-l+1}{s+t-j}\mathbf{1}_{\{j-l+1\leqslant s-k\}}.$$

Similarly (noting that $j \leqslant s - k + t - l$), $\{X \in Q, Y \notin Q\}$ occurs if and only if either (i) $|X \cap S| = s - k$, $y \in S$, and $l \leqslant j - s + k$ or (ii) $|X \cap T| = t - l$, $y \in T$, and $k \leqslant j - t + l$, whence

$$\Pr(X \in Q, Y \notin Q) = \binom{s}{s-k}\binom{t}{j-s+k}\frac{k}{s+t-j}\mathbf{1}_{\{l\leqslant j-s+k\}} + \binom{t}{t-l}\binom{s}{j-t+l}\frac{l}{s+t-j}\mathbf{1}_{\{k\leqslant j-t+l\}}.$$

Thus, since

$$\binom{s}{k-1}(s-k+1) = \binom{s}{s-k}k \quad \text{and} \quad \binom{t}{l-1}(t-l+1) = \binom{t}{t-l}l,$$

we will be done if we show

$$\binom{t}{j-k+1}\mathbf{1}_{\{j-k+1\leqslant t-l\}} \geqslant \binom{t}{j-s+k}\mathbf{1}_{\{l\leqslant j-s+k\}} \tag{18}$$

and

$$\binom{s}{j-l+1}\mathbf{1}_{\{j-l+1\leqslant s-k\}} \geqslant \binom{s}{j-t+l}\mathbf{1}_{\{k\leqslant j-t+l\}}.$$

The easy verifications are similar and we just do (18): we have $j - k + 1 \geqslant j - s + k$ (since $2k \leqslant s$) and $(j - k + 1) + (j - s + k) \leqslant t$ (since $2j \leqslant s + t - 1$), implying both $\binom{t}{j-k+1} \geqslant \binom{t}{j-s+k}$ and $\mathbf{1}_{\{j-k+1\leqslant t-l\}} \geqslant \mathbf{1}_{\{l\leqslant j-s+k\}}$. $\square$

5. Consequences for Mason's conjecture

In this section we prove Theorem 8. As noted at the end of Section 1, this with Theorem 10 (or, for part (a), Theorem 2) immediately implies Theorem 9. Here we do assume a (very) few matroid basics—again, [20] and [15] are standard references—and now denote matroids by M. Our argument mainly follows that of [4], which, as mentioned in Section 1, makes some progress on the "LC version" of Mason's conjecture.

Given a matroid M on ground set E, let $\Pi_i = \Pi_i(M)$ be the set of ordered partitions (A, B) of E with $|A| = i$ and each of A, B independent. Notice that when $|E| = 2k$, APP for uniform measure on the independent sets of M is the inequality $|\Pi_{k-1}| \leqslant \frac{k}{k+1}|\Pi_k|$.

Dowling's point of departure was the observation that if $|\Pi_k(M)| \geqslant |\Pi_{k-1}(M)|$ for every $k \leqslant t$ and every M on an E of size $2k$, then for an arbitrary M (on a ground set of any size) the initial portion $(a_0, \dots, a_{t+1})$ of the sequence of independence numbers is LC. This is, of course, analogous to Theorem 10. Note, though, that, in contrast to Theorem 10, the implication here is quite straightforward;

namely, a natural (and standard) grouping of terms represents the expansion of $a_k^2 \geqslant a_{k-1}a_{k+1}$ as a positive combination of inequalities $|\Pi_k(M)| \geqslant |\Pi_{k-1}(M)|$ for various M's. (If, in analogy with (1), we set $\beta_i(\nu) = \sum\{\nu(\eta)\nu(\underline{1}-\eta)\colon\ \eta \in \Omega, |\eta| = i\}$, then Dowling's argument shows that $\mu \in \mathcal{M}$ is LC provided each ν obtained from μ by conditioning on the values of some $n-2k$ variables satisfies $\beta_k(\nu) \geqslant \beta_{k-1}(\nu)$.)

Dowling also showed that every matroid on a ground set of size $2k \leqslant 14$ satisfies $|\Pi_k| \geqslant |\Pi_{k-1}|$ (which yields the result mentioned in Section 1). This is mainly based on Lemma 18 below and (a version of) the following easy observation, in which we use d for degree and "$\sim$" for adjacency.

Lemma 17. *Let G be a simple, bipartite graph with bipartition $X \cup Y$. If $d(x) \geqslant 1$ for all $x \in X$ and $\sum_{x\sim y} d(x)^{-1} \leqslant C$ for all $y \in Y$, then $|X| \leqslant C|Y|$.*

Proof. This is standard: $|X| = \sum_{x\in X}\sum_{y\sim x} d(x)^{-1} = \sum_{y\in Y}\sum_{x\sim y} d(x)^{-1} \leqslant C|Y|$. □

Proof of Theorem 8. Since the class of measures in question is closed under conditioning, it's enough to show that every matroid M on a ground set E of size $2k \leqslant 10$ satisfies

$$\left|\Pi_{k-1}(M)\right| \leqslant \frac{k}{k+1}\left|\Pi_k(M)\right|. \tag{19}$$

This is trivial when $k = 1$, so we assume $k \in \{2, 3, 4, 5\}$. Define bipartite graphs G_1, G_2 with the common bipartition $\Pi_{k-1} \cup \Pi_k$ by setting, for $(C, D) \in \Pi_{k-1}$ and $(A, B) \in \Pi_k$, $(C, D) \sim (A, B)$ in G_1 (resp. G_2) if $C \subseteq A$ (resp. $C \subseteq B$). Let $G = G_1 \cup G_2$. Then, writing r for rank and d_i and d for degrees in G_i and G, we have (see [4], pp. 24–27)

Lemma 18. *If $r(M) \geqslant k+2$ or $r(M) = k+1$ and M has no coloops, then*

(a) *every $(A, B) \in \Pi_k$ satisfies $2 \leqslant d_i(A, B) \leqslant k$ for $i = 1, 2$;*
(b) *every $(A, B) \in \Pi_k$ satisfies*

$$\sum_{(C,D)\sim(A,B)} \frac{1}{d(C,D)} \leqslant \frac{1}{2}\left(\frac{d_1(A,B)}{d_2(A,B)+1} + \frac{d_2(A,B)}{d_1(A,B)+1}\right);$$

(c) *every $(A, B) \in \Pi_k$ with $d_1(A, B) < d_2(A, B)$ satisfies*

$$\sum_{(C,D)\sim(A,B)} \frac{1}{d(C,D)} \leqslant \frac{1}{2}\left(\frac{d_1(A,B)-1}{d_1(A,B)+1} + \frac{d_2(A,B)-d_1(A,B)+1}{d_1(A,B)+2} + \frac{d_1(A,B)}{d_2(A,B)+1}\right). \tag{20}$$

Proof of (19). We may assume $r(M) > k$ since otherwise $\Pi_{k-1} = \emptyset$. Also, if $r(M) = k+1$ and M has a coloop e, then $|\Pi_{k-1}(M)| = |\Pi_{k-1}(M\setminus e)|$ (since every basis contains e) and $|\Pi_k(M)| = 2|\Pi_{k-1}(M\setminus e)|$, so we have (19).

So we may assume we are in the situation of Lemma 18 (either $r(M) \geqslant k+2$ or $r(M) = k+1$ and M has no coloops). By Lemma 17, it suffices to show that for each $(A, B) \in \Pi_k$,

$$\sum_{(C,D)\sim(A,B)} \frac{1}{d(C,D)} \leqslant \frac{k}{k+1}. \tag{21}$$

Since $d_1(A, B) = d_2(B, A)$, we may assume, using Lemma 18(a), that $2 \leqslant d_1(A, B) \leqslant d_2(A, B) \leqslant k$. If $d_1(A, B) = d_2(A, B)$, then Lemma 18(b) bounds the left side of (21) by $d_1(A, B)/(d_1(A, B) + 1) \leqslant k/(k + 1)$. Otherwise (i.e. if $d_1(A, B) < d_2(A, B)$), Lemma 18(c) bounds the left side of (21) by the right side of (20), which a little calculation shows—this is where we use $k \leqslant 5$—to be at most $d_2(A, B)/(d_2(A, B) + 1) \leqslant k/(k + 1)$. □

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